Distributed primality proving and the primality of (2 +1)/3

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Rapports de Recherche

N° 1152

Programme 1
Programmation, Calcul Symbolique
et Intelligence Artificielle

DISTRIBUTED PRIMALITY PROVING AND THE PRIMALITY OF $(2^{3539} + 1) / 3$

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Décembre 1989

RR-1152

DISTRIBUTED PRIMALITY PROVING AND THE PRIMALITY OF $(2^{3539}+1)/3$

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Abstract. The aim of this report is to explain the implementation of the Elliptic Curve Primality Proving algorithm of Atkin in a distributed way. We describe the use of a `Le_Lisp` program managed by a bunch of script-shells that run processes on different workstations communicating via the distributed file system NFS. As a result, the primality of 700-digit numbers can be done routinely in about a week of CPU on 10 SUN's. Using twelve SUN's and one month and a half CPU, the author was able to prove the primality of a very large number (1065 digits): This is (up to now) the record for the largest number ever tested for primality by a general purpose algorithm.

PRIMALITE DISTRIBUEE ET LA PRIMALITE DE $(2^{3539}+1)/3$

Résumé. Nous expliquons dans ce rapport comment on peut implémenter de façon distribuée l'algorithme de Preuve de Primalité d'Atkin utilisant les courbes elliptiques. Nous décrivons l'emploi d'un programme `Le_Lisp` géré par une série de script-shells qui pilotent différentes stations de travail via le système de gestion de fichiers distants NFS. De cette façon, il est aujourd'hui possible de tester la primalité de nombres de 700 chiffres de manière routinière en une semaine de temps CPU sur 10 SUNs. Utilisant 12 stations et un mois et demi de temps CPU, l'auteur a résussi à prouver la primalité d'un nombre de 1065 chiffres. Il détient ainsi actuellement le record du plus grand nombre déclaré premier par un programme général.

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DISTRIBUTED PRIMALITY PROVING AND THE PRIMALITY OF $(2^{3539}+1)/3$

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PREPRINT

December 12, 1989

Abstract

We explain how the Elliptic Curve Primality Proving of Atkin can be implemented in a distributed way. Applications are given to the certification of large primes (more than 500 digits). As a result, we describe the successful attempt at proving the primality of the 1065-digit $(2^{3539}+1)/3$, the first *ordinary* Titanic prime.

1 Introduction

For cryptographical purposes [7], it is desirable to generate large primes as fast as possible. This can be done via *ad hoc* techniques [31, 13, 15, 4] or by means of a general purpose primality testing algorithm such as that described in [1, 12, 11, 6] or the Elliptic Curve Primality Proving (ECPP) algorithm of Atkin [2, 27, 25] (For a survey of primality testing, see [19]).

Another point is to certify large primes, such as the Cunningham numbers [9], which sometimes have more than 400 digits. The purpose of this paper is to explain how the ECPP algorithm has been implemented on a network of SUN workstations and used to test some numbers with more than 500 digits for primality.

We first begin by a short introduction to ECPP and then, we explain the distributed process *à la* Lenstra-Manasse [20]. We also give the history of the primality of the breaking-record $\mathcal{N}_{3539} = (2^{3539}+1)/3$.

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2 A brief description of ECPP

2.1 Elliptic curves

Let $\mathbf{K}$ be a field of characteristic prime to 6. An elliptic curve E over $\mathbf{K}$ is a non singular algebraic projective curve of genus 1. It can be shown [10, 34] that E is isomorphic to a curve with equation:

$$y^2z = x^3 + axz^2 + bz^3, \tag{1}$$

with a and b in $\mathbf{K}$. The *discriminant* of E is $\Delta = -16(4a^3 + 27b^2)$ and the *invariant* is

$$j = 2^8 3^3 \frac{a^3}{4a^3 + 27b^2}.$$

We write $E(\mathbf{K})$ for the set of points with coordinates $(x : y : z)$ which satisfy (1) with $z = 1$, together with the point at infinity: $O_E = (0 : 1 : 0)$. We will use the well-known *tangent-and-chord* addition law on a cubic [17] over a finite field $\mathbf{Z}/N\mathbf{Z}$ (see [22] for a justification).

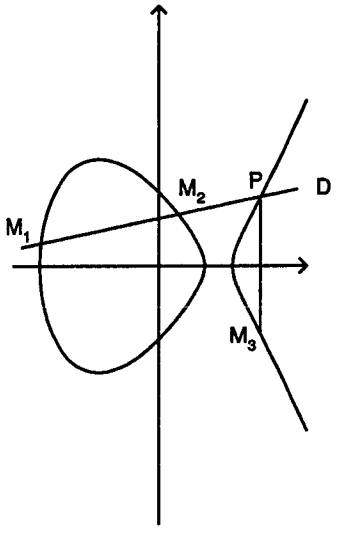


Figure 1: An elliptic curve over $\mathbf{R}$.

In order to add two points $M_1 = (x_1, y_1)$ and $M_2 = (x_2, y_2)$ on E resulting in $M_3 = (x_3, y_3)$, the equations are

$$\begin{cases} x_3 &= \lambda^2 - x_1 - x_2 \\ y_3 &= \lambda(x_1 - x_3) - y_1 \end{cases}$$

where

$$\lambda = \begin{cases} (y_2 - y_1)(x_2 - x_1)^{-1} & \text{if } x_2 \neq x_1 \\ (3x_1^2 + a)(2y_1)^{-1} & \text{otherwise} \end{cases}$$

We can compute kP using the binary method [18] (see also [11]) or addition-subtraction chains [30].

2.2 Primality testing

Let us recall one of the converses of Fermat's theorem.

Theorem 1 ([32]) *Let s be a divisor of $N-1$. Let a be an integer prime to N such that*

$$a^{N-1} \equiv 1 \bmod N \text{ and } \gcd(a^{(N-1)/q} - 1, N) = 1,$$

for each prime divisor q of s. Then each prime divisor p of N satisfies: $p \equiv 1 \bmod s$.

Corollary 1 *If $s > \sqrt{N} - 1$ then N is prime.*

A similar theorem can be stated for elliptic curves.

Theorem 2 ([14, 21]) *Let N be an integer greater than 1 and prime to 6. Let E be an elliptic curve over $\mathbf{Z}/N\mathbf{Z}$, m and s two integers such that $s \mid m$. Suppose we have found a point P on E that satisfies $mP = O_E$, and that for each prime factor q of s, we have verified that $\frac{m}{q}P \neq O_E$. Then if p is a prime divisor of N, $\#E(Z/pZ) \equiv 0 \bmod s$.*

Corollary 2 *If $s > (\sqrt[4]{N} + 1)^2$, then N is prime.*

In order to use the preceding theorem, we need to compute the number of points m. This process is far from trivial in general (see [33]). From a practical point of view, it is desirable to use deep properties of elliptic curves over finite fields. This involves the theory of complex multiplication and class fields and requires a lot of theory [27]. We can summarize the principal properties:

Theorem 3 *Every elliptic curve $E \bmod p$ has complex multiplication by the ring of integers of an imaginary quadratic field $K = \mathbf{Q}(\sqrt{-D})$.*

From a very down-to-earth point of view, this comes down to saying:

- p splits completely in K: $(p) = (\pi)\,(\pi')$ in K;
- $H_D(j(E)) \equiv 0 \bmod p$ for a fixed $H_D(X)$ in $\mathbf{Z}[X]$;
- $m = (\pi - 1)(\pi' - 1) = p + 1 - t$, where $|t| \leq 2\sqrt{p}$ (Hasse).

The computation of the polynomials H_D is dealt with in [27] and [28].

2.3 Outline of ECPP

We now explain how the preceding theorems are used in a *factor and conquer* algorithm similar to the DOWNRUN process of [35]. The first phase of the algorithm consists in finding a sequence $N_0 = N > N_1 > \cdots > N_k$ of probable primes such that: N_{i+1} prime $\Longrightarrow N_i$ prime. The second then proves that each number is prime, starting from N_k.

Procedure SearchN

1. $i := 0$; $N_0 := N$;
2. find a fundamental discriminant $-D$ such that (N_i) splits as the product of two principal ideals in $\mathbf{Q}(\sqrt{-D})$;

3. for each solution of $(N_i) = (\pi)(\pi')$, find all factors of $m_\pi = (\pi - 1)(\pi' - 1)$ less than a given bound B and let N_π be the corresponding cofactor;

4. **if** one of the N_π is a probable prime **then** set $N_{i+1} := N_\pi$, store $\{N_i, D, \pi, m_\pi\}$ set $i := i+1$, and go to step 2 **else** go to step 3.

5. **end.**

The second phase consists in proving that the numbers N_i are indeed primes. This is done as follows:

Procedure Proof

for $i = k..0$

1. compute a root j of $H_{D_i}(X)$ mod N_i as described in [28, 29];

2. find an equation of the curve E_i whose invariant is j and cardinality m_i;

3. verify the condition of theorem (2).

end.

For more details, the reader is referred to [2].

3 Large primes

The author used ECPP to test a lot of numbers from the Cunningham tables [9] and some others, namely $S_p = ((1+\sqrt{2})^p + (1-\sqrt{2})^p)/2$ for $p \in \{1493, 1901\}$ with respectively 572 and 728 digits, in 30 and 40 days on a single workstation. Indeed, a simple extrapolation shows that testing a 1000-digit number would require about 6 months (at least). We must do something else to increase the bound on the largest number ECPP can test.

4 Distributed computations

From the preceding description, it is easy to see that this algorithm is very well suited for distributed computations. We can do the first phase in parallel and then the second one too. Let us see how I did this.

First of all, I implemented ECPP using the `Le_Lisp` language and the multiprecision described in [16]. Then the computations were done using a *master* ($\mathfrak{M}$) and an indefinite number of SUN workstations, called *slaves* ($\mathfrak{S}$). The idea is that when dealing with very large numbers, the crucial part of ECPP is the first one, because it requires the factorization of very large numbers.

There are basically two ways of doing that. The first one is to try to factor a single number using all the stations. The second is to let each station work on a different number. Actually, I use the latter scheme, because the first one would require more communications and also because it is not the *right* philosophy of the test: The less factoring power we use, the better.

We now describe the conditions required to do an optimal job.

4.1 Constraints

We want to use the idle time of a network of SUN's. We do this in a way similar to that of [20]. We start a process on a machine in such a way that a legitimate user is not (too much) disturbed: If a user types on a console (in UNIX words, he changes the date of one of the `tty`'s), then the program is stopped (by means of a `kill -STOP`) and restarted 10 minutes after the last action of the user (with a `kill -CONT`). The process is also stopped whenever the load climbs up some prescribed value (typically 1.5) and is subjected to the same restart conditions. All this is done with the script shells distributed by Mark Manasse for integer factorization. Another important feature of these programs is the ability to restart themselves after a small crash such as a `Connection timed out` from a server. Also, they do not depend on a particular machine (at least running UNIX or ULTRIX) or a particular language. It is possible to use a `C` program on a DEC station and a `Le_Lisp` program on a SUN.

4.2 The first phase

4.2.1 Role of the master

On $\mathfrak{M}$ (typically the author's own workstation), the program used does the following things for each N_i of the first phase:

1. put in the file WHICHN the number to be tested;
2. find all fundamental discriminants D (from a finite subset $\mathcal{D}$) for which N_i is represented by a form of G_0 and put them in the file DSET;
3. initialize the rank of the next D to be examined to 1 in the file DRANK;
4. start finding a suitable D.

4.2.2 Role of the slaves

On $\mathfrak{S}$, the program looks like:

1. read the number to be tested from WHICHN and call it N;
2. while N is equal to the content of WHICHN, select a new D in DSET, update DRANK and try to factor any of the m_π.

4.2.3 Tasks performed by every machine

Each machine does the following:

1. find a D such that (N) splits completely in $\mathbf{Q}(\sqrt{-D})$;
2. try to factor each m_π using first trial division, then Pollard's ρ method, then the $p-1$ method and finally ECM.

Inside each factoring algorithm, the program periodically tests whether something has happened. When this is so, it gives up on N_i and enters a new work on N_{i+1}. When using the ρ method [24], the test is done at each gcd (for our purposes, there are 10^4 iterations and a gcd each 1000 iterations). During $p-1$, only once. In the ECM method using the parallel affine scheme [8, 24], each time an elementary operation has been done (this is because such an operation is quite expensive).

4.2.4 Communications between $\mathfrak{M}$ and $\mathfrak{S}$

The files DSET, DRANK and WHICHN have just been described. All this supposes the use of a distributed file system: here it is NFS that does all the job. Special code has been written to handle the problems arising when one machine wants to read a file while another tries to write in it or to test whether the file can be accessed through NFS.

4.3 The second phase

For each N_i, it remains to check the primality conditions. Using a file containing the next number to be certified, each station takes the useful data and does its job. It should be noted that this phase can be started even if the first one is not complete: However, there must be something to prove...

4.4 Problems encountered

One of the major problem is the reliability of the NFS protocole, especially when using machines not depending from the same file server. The program is very well suited for testing the reliability of the network. Each time there is a connection problem, the process simply crashes.

Also, using a `Le_Lisp` executable requires a lot of memory and, sometimes, this resulted in a swap problem and also a crash.

5 History of $\mathcal{N}_{3539}$

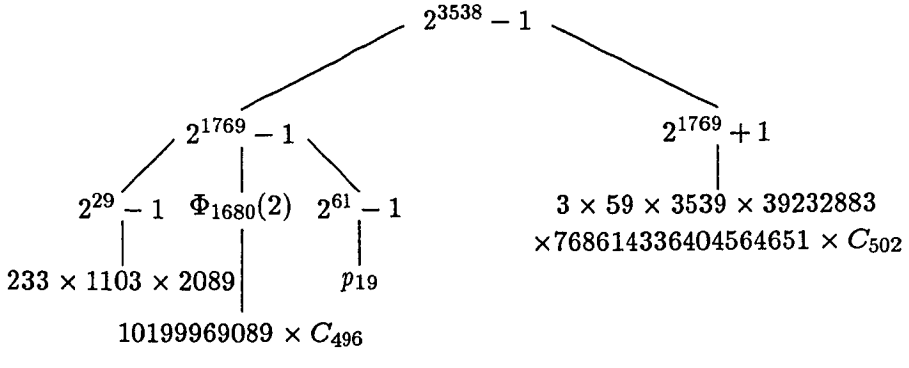
5.1 Entomology of a Record

During their setting of the new Mersenne's conjecture [3], the authors tested some numbers of the form $\mathcal{N}_p = (2^p + 1)/3$ for primality. They found that $\mathcal{N}_p$ was a probable prime for $p \in \{1709, 2617, 3539\}$.

During EUROCRYPT '89 (April 10-13, 1989), it appeared that both ECPP and the Jacobi Sums test [12, 11, 6] were able to attack numbers as large as 1000 digits. This was the very start of a stimulating competition with W. Bosma and M.-P. van der Hulst.

Indeed, the first of these numbers ($p = 1709$, $\mathcal{N}_p$ with 514 digits) was the first number proven prime using ECPP in its distributed version. This was done on April 19, 1989 with three SUN's and four days of CPU.

Then, I decided to skip $p = 2617$ and try $\mathcal{N}_{3539}$. As shown by in the following figure, the factorization of $\mathcal{N}_{3539} - 1$ is not complete (up to now). Anyway, it was more fun to use ECPP.



Using ECPP, I first launch the process on April 20, 1989. The set $\mathcal{D}$ mentionned above consisted of all D's with $h(-D) \leq 20$, sorted according to $(h/g, h, D)$. Following [2], the *difficulty* of testing N may be defined as

$$\Phi(N) = e^{-\gamma} \frac{\log N}{M(N)}$$

where γ is Euler's constant and $M(N)$ is defined as follows. Put

$$M(N) = \sum_{\substack{D \in \mathcal{D} \\ N \in G_0(-D)}} w(-D) \frac{g(-D)}{h(-D)},$$

where the summation is on all D for which N can be represented by a form of the principal genus of quadratic forms of discriminant $-D$, $g(-D) = 2^{t-1}$ with t the number of prime factors of D and $h(-D)$ the class number. As a matter of fact, $\Phi(N)$ yields the value $\log B$ of the upper bound on the largest factor of a number of points m we must factor in order to find a good candidate.

Coming back to $\mathcal{N}_{3539}$, I found that $M(\mathcal{N}_{3539}) = 55$ yielding $\log_{10} B = 11$. This implied in turn that the only way to achieve this was using ECM. At that time, I hadn't implemented this and so the program started using only Pollard ρ and the $p-1$ method *. This first attempt lasted till May 13, without any result: I couldn't even find a good N_1. There was something to be done. Moreover, some problems seemed to arise in the $p-1$ method, where the routine seemed to loop forever in some cases.

When looking at

$$\log B = \Phi(N), \tag{2}$$

there are two disctinct ways of solving the problem. The first one is to use sophisticated factoring routines, the other one is to increase the value of $M(N)$. I used the second and decided to enlarge $\mathcal{D}$ with all D less than 2^{15} †. This increased $M(N)$ to the value of 174, yielding $\log_{10} B = 3.44$. This clearly said that ECM was no more necessary and that ρ was enough. After fixing some stupid bug in my ρ routine, I re-started the program on June 5 and it lasted till July 10, yet without any result.

Clearly, there was a problem. Using induction, it seemed clear that there was, somewhere, a deep bug that only appeared when dealing with large numbers, but not with small ones. So I decided to stop working on $\mathcal{N}_{3539}$, and began to reassure myself with a smaller one, namely $\mathcal{N}_{2617}$ (788 digits). Although this number could have been done by simply factoring $\mathcal{N}_{2617} - 1$ (as remarked by Atkin), this attempt was designed to find this bug. So, the process started on July 21 and ended on August 19, proving the number to be prime, but without revealing any bug.

At this point, I decided to implement ECM, just to see if something would happen to change. I had problems with this, since it was only possible to use the first phase of the algorithm, all the second phases requiring too much memory (they all need about $(\log N)^2$ storage, making it infeasible for 1000-digit numbers). Moreover, I could only use 20 curves or so, again because the storage was making it prohibitive to use on workstations with not too much memory, such as a standard SUN 3/50 (4 Mo). This was quite a disappointment. The third attempt on $\mathcal{N}_{3539}$ was then started on September 12 and took two weeks. Nothing observable happened.

I decided then to replace all these D's less than 2^{15} with all D's with h/g small, irrespective of the size of D as soon as D fitted in a 32-bit word. More precisely, I computed and stored all

*As suggested by Atkin, $p-1$ is worth using when dealing with Cunningham numbers, because they have non-trivial arithmetic properties.

†This limitation comes from the language I used, Le_Lisp, which does not accept 32-bit integers.

D with $h(-D) \leq 50$ (plus some with $h = 64$) and ordered them according to $(h/g, h, D)$. This yielded $M(\mathcal{N}_{3539}) = 291$ and $\log_{10} B = 2.05$. What would appear as the last attempt then began on September 29.

A feeling of deep personal gratification came over me when, on October 5, 1989, I finally confirmed my initial impression that a part of the program was irretrievably bug ridden. This occurred when I thoroughly checked my factoring routines. I had simply forgotten to reduce the parameters of $\rho, p-1, \ldots$ after a factor was discovered ! The program was thus *asymptotically bugged*: When dealing with small numbers, I need maybe one large factor, but with large numbers, maybe two or more. This explained also the above mentionned problem with $p-1$ (because of the way the exponentiation routine was programmed, it wanted to find the first 1 in the binary expansion of a zero word).

And (not surprisingly ?), it began to work. The breakthrough occured on October 6, when N_1 was reached, using $D = 97507$ (with $h = 36, g = 2$). After that, this was quite a quiet work, except that there happened to be a difficult client at one stage (namely N_{11}), requiring a D with $h = 56$ and $g = 2$. The building of the tower of primes was finally completed on November 8 at 830 pm (Inria-Paris time).

Meantime, I had used one workstation for the second part of the process, proving the numbers to be primes. One week before the end of the first phase, I also used one of the SUN's on this $h/g = 28$ business, that is finding a root of a polynomial of degree 28 over a finite field with about 10^{991} elements. For that, I chose the most resistant SUN I could find. By this, I mean a station that was able to resist to all network problems that could appear. Actually, this was a period of time where there was quite a lot of those. This computation took one week.

When the first phase ended, about 40 proving steps were done and I was able to launch the workstations on the remaining cases. On November 11, it was over, even the 28 degree stuff. It was it, I had sunk the Titanic, this time with an *ordinary* prime (as opposed to the Elliptic Mersenne Primes of [26]): The problem of testing 1000-digit numbers for primality was solved. Looking at the whole story, it took only one month and a half to do that. Moreover, it took only one week to come down from 700 digits to 10: This means that one can routinely test such numbers for primality. Some further experiments are being done to verify this.

The final result is a file of 500 kbytes consisting of the certificate of primality for $\mathcal{N}_{3539}$. This file can be sent to anyone who wants to check it using the protocole described in [27].

5.2 Technical details

In order to prove the primality of $\mathcal{N}_{3539}$, I used 12 SUN workstations, among which four 3/50, seven 3/60 and one 3/160 with a special chip designed for 512-bit multiplication [5]. Using the full power of the chip was done by using Montgomery's ideas on modular multiplications [23]: These ideas were only used for Pollard ρ, $p-1$ and pseudoprimality tests. The speedup for a modular exponentiation of 110 words of 32 bits is about 8.

The first phase took approximately 288 days of CPU (only one month and a half in human time). The second one 31 days of CPU. The total time is thus less than one year of CPU. The tower of primes consists of 162 numbers. In Figure 2, we print the number of digits of N_i versus the human time from the start of the job. The distribution of the *gains*, that is the number of digits we win in finding the following member is displaied in Figure 3: the mean value is 6.5, with minimum 0 and maximum 34. In Figure 4, we put the distribution of the values of h/g, the mean value being 3.49. Finally, Figure 5, we printed the value of $M(N_i)$ versus the time needed to reach N_{i+1}: This looks very odd.

6 Conclusions

We see that ECPP in its distributed implementation is a very powerful tool to test arbitrary large numbers for primality. It should be able to deal with somewhat larger numbers (maybe with 1200 digits or so). The problem that is bound to arise is that there is a point where we need powerful factoring routines such as ECM. However, this would slow down the running time of the whole process. So it seems not possible to deal with 2000-digit numbers.

Another interesting point is to wait for the results of van der Hulst and Bosma on the Jacobi Sums program. Since they do not depend on integer factoring, it may be faster.

Acknowledgments. First of all, I thank the owners of the workstations that contributed to my record, namely L. Albert, L. Audoire, J.-J. Codani, V. Collette, P. Flajolet, P. Jacquet, P. Le Chenadec, M. Régnier, B. Serpette and P. Zimmermann. W. Bosma and M.-P. van der Hulst (and also A. K. Lenstra) must be thanked for their stimulating work to attack the supremacy of ECPP using the Jacobi Sums test. Without the script-shells of M. Manasse, this job would have been less easy: special thanks to him, then. Thanks to R. Ehrlich who helped me modifying the above scripts and explained to me some of the magic properties of NFS. Thanks also to I. Vardi for (helpful or stylistic) comments about my manuscript.

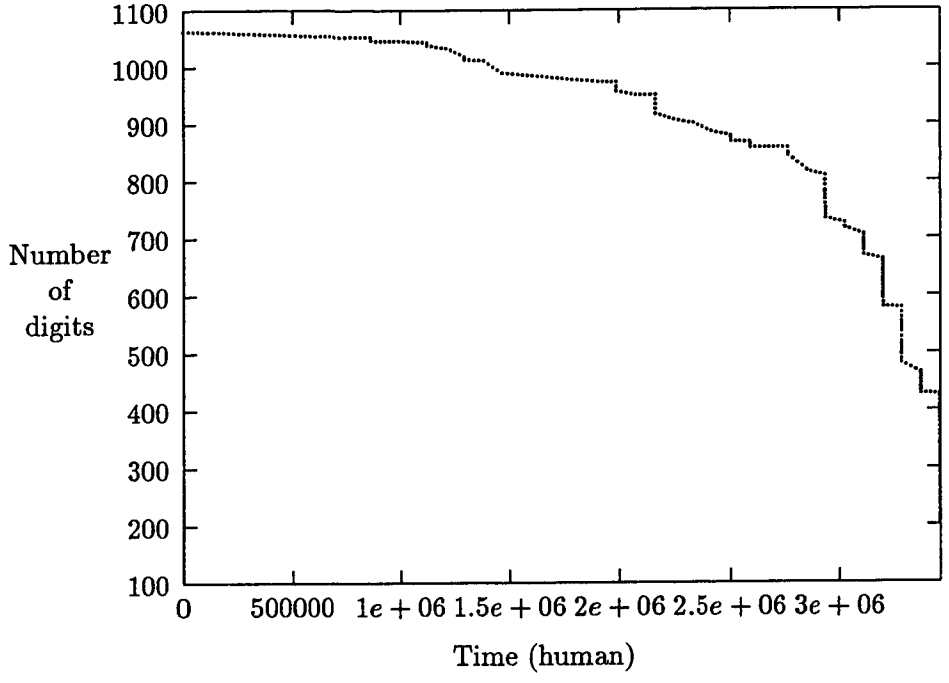


Figure 2: Number of digits reached vs. human time

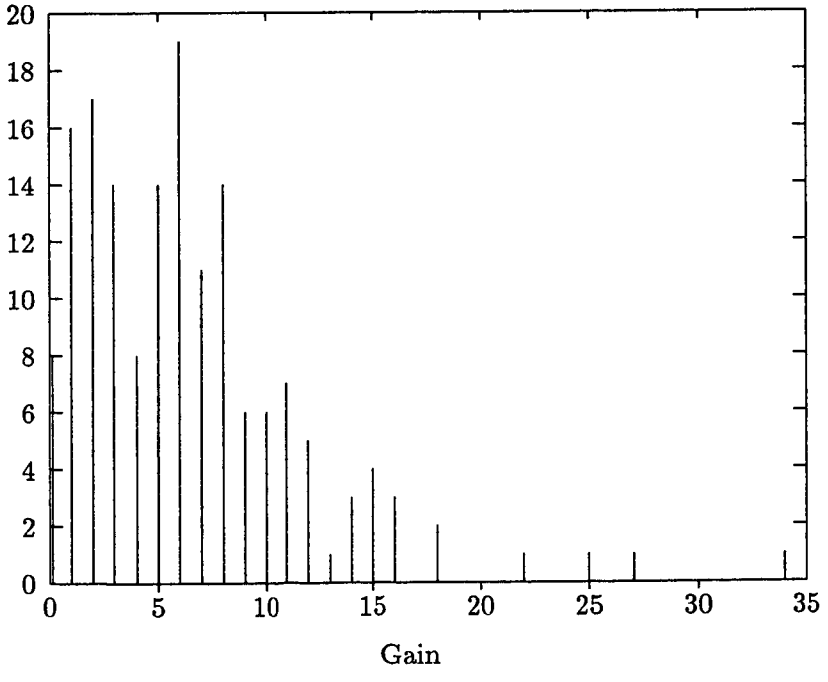


Figure 3: Number of digits gained at each step

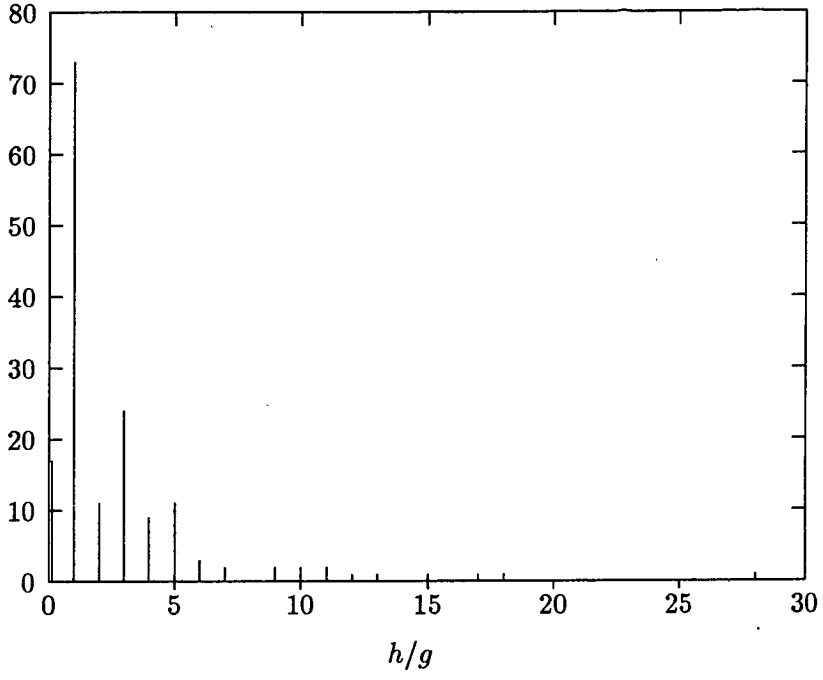


Figure 4: Distribution of h/g

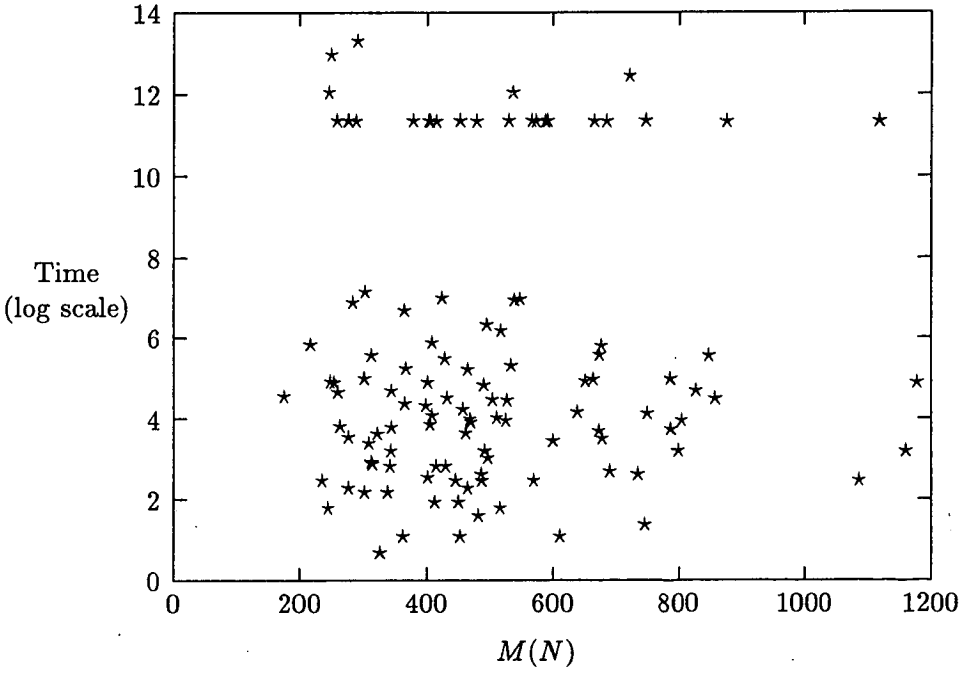


Figure 5: Difficulty vs. time needed for the next step

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Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique